Equation of Motion of Relativistic Electron in Stationary Axially Symmetric Electric and Magnetic Fields under Forces of arbitrary Nature

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Abstract – Apart from the action of electric and magnetic fields, axially symmetric motion of electron subjected to forces of another character has been considered. Parametric system of equation (parameter – time), explicitly independent of outside forces has been obtained. The effect of the outside forces is taken into account through a variation in energy and angular momentum of electron. By way of elimination of time the system is reduced to the equation of trajectory.

The equation of motion of relativistic electron has the form:

$$\frac{d}{dt}\left(\gamma m_0 \vec{V}\right) = e\vec{E} + e\left[\vec{V}\times\vec{B}\right] + \vec{F} \qquad (1)$$

where e, m_0 – the charge and rest mass of particle, $\gamma = \left(1-\beta^2\right)^{-1/2}$, $\beta = v/c$, $\vec{V}$ – the velocity of particle, v – its absolute value, c – velocity of light, $\vec{E}$, $\vec{B}$ – vectors of electric and magnetic intensities generated by external sources, $\vec{F}$ – vector of forces of arbitrary character.

Using relationships

$$\frac{d\gamma}{dt} = \frac{\gamma^3 v}{c^2}\frac{dv}{dt},\ vdv = \vec{V}d\vec{V}$$

we rearrange Eq. (1) to the form:

$$m_0\gamma^3\frac{d\vec{V}}{dt} = e\vec{E} + e\left[\vec{V}\times\vec{B}\right] + \vec{F}.$$

Scalar multiplication of both sides by $\vec{V}$ gives:

$$m_0\gamma^3 v\frac{dv}{dt} = e\left(\vec{E}\vec{V}\right) + \left(\vec{F}\vec{V}\right)$$

or

$$m_0 c^2\frac{d\gamma}{dt} = e\left(\vec{E}\vec{V}\right) + \left(\vec{F}\vec{V}\right) \qquad (2)$$

Since in the stationary case we have

$$\frac{d\gamma}{dt} = \vec{V}grad\gamma,$$

then, as a consequence of the correctness of Eq.(2), for an arbitrary $\vec{V}$, we obtain:

$$grad\gamma = \frac{e\vec{E}+\vec{F}}{m_0 c^2}. \qquad (3)$$

The starting equation (1) takes the form:

$$\frac{d}{dt}\left(\gamma\vec{V}\right) = c^2 grad\gamma + \frac{e}{m_0}\left[\vec{V}\times\vec{B}\right] \qquad (4)$$

In cylindric coordinates

$$\vec{V} = \left(\dot{r}, r\dot{\theta}, \dot{z}\right),\ \vec{E} = \left(E_r, E_\theta, E_z\right),$$
$$\vec{B} = \left(B_r, B_\theta, B_z\right),\ \vec{F} = \left(F_r, F_\theta, F_z\right)$$

and Eq. (4) has the componentwise form:

$$\begin{cases} \frac{d}{dt}\left(\gamma\dot{r}\right) - \gamma r\dot{\theta}^2 = c^2\frac{\partial\gamma}{\partial r} + \frac{e}{m_0}\left(r\dot{\theta}B_z - B_\theta\dot{z}\right) \\ \frac{1}{r}\frac{d}{dt}\left(\gamma r^2\dot{\theta}\right) = c^2\frac{\partial\gamma}{\partial\theta} + \frac{e}{m_0}\left(\dot{z}B_r - \dot{r}B_z\right) \\ \frac{d}{dt}\left(\gamma\dot{z}\right) = c^2\frac{\partial\gamma}{\partial z} + \frac{e}{m}\left(\dot{r}B_\theta - r\dot{\theta}B_r\right) \end{cases} \qquad (5)$$

In the case of axially-symmetric problems, it is felt [1] that

$$E_\theta = 0,\ \frac{\partial\gamma}{\partial\theta} = \frac{F_\theta}{m_0},$$
$$B_z = \frac{1}{2\pi r}\frac{\partial\Psi}{\partial r},\ B_r = \frac{-1}{2\pi r}\frac{\partial\Psi}{\partial z}, \qquad (6)$$

where F_θ – the azimuth component, Ψ – the magnetic field flux

$$\Psi(r,z) = 2\pi\int_0^r rB_z(r,z)dr. \qquad (7)$$

The second equation of the system (5) can be written as follows:

$$\frac{1}{r}\frac{d}{dt}\left(\gamma r^2\dot{\theta}\right) = -\frac{e}{2\pi m_0 r}\frac{d\Psi}{dt} + \frac{F_\theta}{m_0}, \qquad (8)$$

where $\frac{d\Psi}{dt} = \frac{\partial\Psi}{\partial z}\dot{z} + \frac{\partial\Psi}{\partial r}\dot{r}$.

From the mechanics course it is known

$$F_\theta = \frac{1}{r}\frac{dM}{dt}, \qquad (9)$$

the angular momentum of outside forces $\vec{F}$ about the beam axis.

Substitute Eq. (9) in Eq. (8) and multiply both sides by r and integrate

$$\gamma r^2\dot{\theta} = -\frac{e}{2\pi m_0}\Psi + \frac{M}{m_0} + C. \qquad (10)$$

Here C – a constant of integration which is derived from the conditions in the starting plane:

$$C = \gamma_0 r_0^2 \dot{\theta}_0 + \frac{e}{2\pi m_0}\Psi_0 - \frac{M_0}{m_0}, \qquad (11)$$

where M_0 is defined by outside forces, and $\gamma_0 r_0^2 \dot{\theta}_0$ – current parameters of particle.

Express $\dot{\theta}$ from Eq. (10) and substitute it in the first and third equations of the system (5), then we obtain in terms of Eq. (6):

$$\begin{cases} \frac{d}{dt}(\gamma\dot{r}) = c^2 \frac{\partial\gamma}{\partial r} + \\ \frac{1}{\gamma r^3}\left(-\frac{e}{2\pi m_0}\Psi + \frac{M}{m_0} + C\right)^2 + \\ \frac{e}{\gamma m_0}\frac{1}{2\pi r^2}\frac{\partial\Psi}{\partial r}\left(-\frac{e}{2\pi m_0}\Psi + \frac{M}{m_0} + C\right), \\ \frac{d}{dt}(\gamma\dot{z}) = c^2 \frac{\partial\gamma}{\partial z} + \\ \frac{e}{\gamma m_0}\frac{1}{2\pi r^2}\frac{\partial\Psi}{\partial z}\left(-\frac{e}{2\pi m_0}\Psi + \frac{M}{m_0} + C\right). \end{cases} \qquad (12)$$

If the function

$$g(r,z) = \frac{\gamma^2 c^2}{2} - \frac{1}{2r^2}\left(-\frac{e}{2\pi m_0}\Psi + \frac{M}{m_0} + C\right)^2, \qquad (13)$$

is introduced, the system of equations (12) takes the symmetric form:

$$\begin{cases} \frac{d}{dt}(\gamma\dot{r}) = \frac{1}{\gamma}\frac{\partial g}{\partial r} + \frac{1}{\gamma m_0 r}\frac{\partial M}{\partial r}\sqrt{\gamma^2 c^2 - 2g}, \\ \frac{d}{dt}(\gamma\dot{z}) = \frac{1}{\gamma}\frac{\partial g}{\partial z} + \frac{1}{\gamma m_0 r}\frac{\partial M}{\partial z}\sqrt{\gamma^2 c^2 - 2g}. \end{cases} \qquad (14)$$

and Eq. (10):

$$\dot{\theta} = \frac{1}{r}\sqrt{\gamma^2 c^2 - 2g}. \qquad (15)$$

The system (14) along with Eq. (3), (13) and (15) describe the motion of the charged particle in axially symmetric fields of electric, magnetic and other forces. As differentiated from the system (5), the right-hand side here does not include any terms depending explicitly on time, which makes it more preferable in analysis of the motion in stationary fields.

Using the equation of trajectory allows us to do away fully with time dependence. For this purpose we will write the following relations:

$$\dot{r} = \frac{dr}{dz}\dot{z}, \ddot{r} = \ddot{z}\frac{dr}{dz} + \dot{z}^2\frac{d^2 r}{dz^2},$$
$$\dot{z}^2 + \dot{r}^2 + (r\dot{\theta})^2 = \beta^2 c^2 = \frac{\gamma^2 - 1}{\gamma^2}c^2. \qquad (16)$$

Then

$$\dot{z}^2\left[1 + \left(\frac{dr}{dz}\right)^2\right] = \frac{\gamma^2 - 1}{\gamma^2}c^2 - (r\dot{\theta})^2$$

and in terms of Eq. (10) and (13)

$$\dot{z}^2 = \frac{2g - c^2}{\left[1 + \left(\frac{dr}{dz}\right)^2\right]\gamma^2}, \dot{r}^2 = \frac{2g - c^2}{\left[1 + \left(\frac{dr}{dz}\right)^2\right]\gamma^2}\left(\frac{dr}{dz}\right)^2. \qquad (17)$$

Express the accelerations $\ddot{r}$ and $\ddot{z}$ from Eq. (14) and (15) and substitute them in the second equality (16):

$$-\frac{\dot{\gamma}\dot{r}}{\gamma} + \frac{1}{\gamma^2}\frac{\partial g}{\partial r} + \frac{1}{m_0 r\gamma^2}\sqrt{\gamma^2 c^2 - 2g}\frac{\partial M}{\partial r} =$$
$$\left(-\frac{\dot{\gamma}\dot{z}}{\gamma} + \frac{1}{\gamma^2}\frac{\partial g}{\partial z} + \frac{1}{m_0 r\gamma^2}\sqrt{\gamma^2 c^2 - 2g}\frac{\partial M}{\partial z}\right)\frac{dr}{dz} +$$
$$\frac{2g - c^2}{\left[1 + \left(\frac{dr}{dz}\right)^2\right]\gamma^2}\frac{d^2 r}{dz^2}.$$

After rearrangements we obtain finally:

$$\frac{d^2 r}{dz^2} = \frac{1 + \left(\frac{dr}{dz}\right)^2}{2g - c^2}\left[\frac{\partial g}{\partial r} - \frac{\partial g}{\partial z}\frac{dr}{dz} + \right.$$
$$\left.\frac{1}{m_0 r}\sqrt{\gamma^2 c^2 - 2g}\left(\frac{\partial M}{\partial r} - \frac{\partial M}{\partial z}\frac{dr}{dz}\right)\right]. \qquad (18)$$

This equation has to be supplemented with the relations (3), (7), (9), (11) and (13), as well as with initial conditions for electron in the starting plane: $r = r_0$, $dr/dz = r_0'$.

References

[1] S.I. Molokovskiy, A.D. Sushkov, *Intense electron and ion beams.* Moscow, Energoatomizdat, 1991.